Warsaw University of Technology

Faculty of Power and Aeronautical Engineering

WARSAW UNIVERSITY OF TECHNOLOGY

Institute of Aeronautics and Applied Mechanics

Finite element method 2 (FEM 2)

Mass matrix of a beam

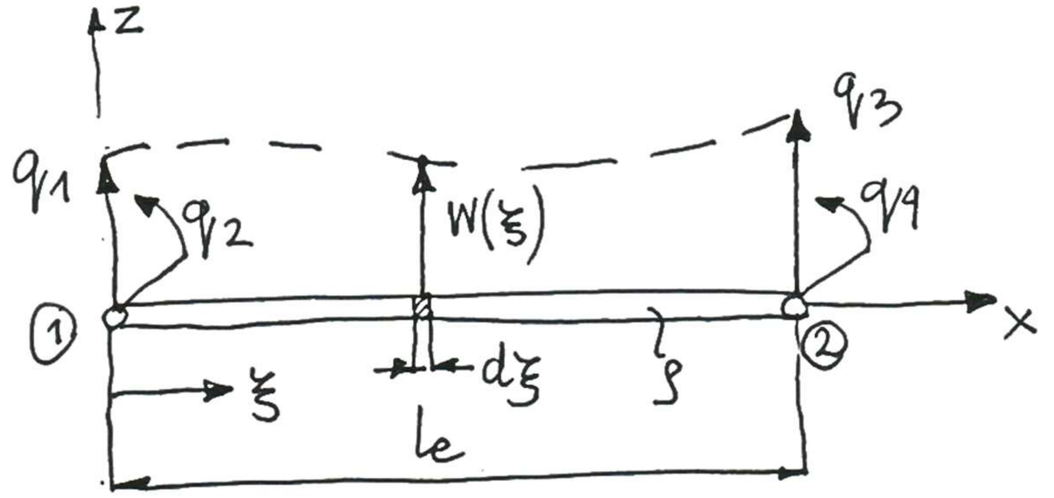
11.2021

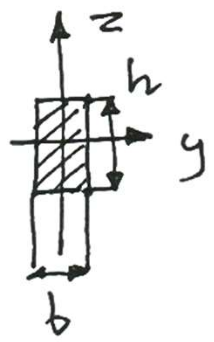
MASS MATRIX OF A BEAM ELEMENT

$n = 2$

$n_p = 2$

$n_e = n \cdot n_p = 4$





$A = b \cdot h$

$J_y = \dfrac{bh^3}{12}$

(rectangular cross-section)

$$\underset{1\times 4}{\lfloor q \rfloor} = \lfloor q_1, q_2, q_3, q_4 \rfloor$$

Kinetic energy of a small part:

$$dT_e = \frac{1}{2} dm\, \dot{w}^2 = \frac{1}{2} \varrho A \, d\xi \, \dot{w}^2 \quad , \dot{w} = \frac{dw}{dt}$$

Kinetic energy of a finite element:

$$T_e = \int dT_e = \frac{1}{2}\int_0^{l_e} \underbrace{\dot{w}}_{\underset{1\times 4}{\lfloor \dot{q} \rfloor_e} \cdot \begin{Bmatrix} N_1(\xi) \\ N_2(\xi) \\ N_3(\xi) \\ N_4(\xi) \end{Bmatrix}} \cdot \varrho A \cdot \underbrace{\dot{w}}_{\lfloor N_1, N_2, N_3, N_4 \rfloor \cdot \underset{4\times 1}{\{\dot{q}\}_e}} \, d\xi \qquad =$$

$$= \frac{1}{2} \underset{1\times 4}{\lfloor \dot{q} \rfloor_e} \cdot \int_0^{l_e} \varrho A \begin{bmatrix} N_1 \cdot N_1 & N_1 \cdot N_2 & N_1 \cdot N_3 & N_1 \cdot N_4 \\ N_2 \cdot N_1 & N_2 \cdot N_2 & N_2 \cdot N_3 & N_2 \cdot N_4 \\ N_3 \cdot N_1 & N_3 \cdot N_2 & N_3 \cdot N_3 & N_3 \cdot N_4 \\ N_4 \cdot N_1 & N_4 \cdot N_2 & N_4 \cdot N_3 & N_4 \cdot N_4 \end{bmatrix} d\xi \cdot \underset{4\times 1}{\{\dot{q}\}_e} \Longrightarrow$$

where:

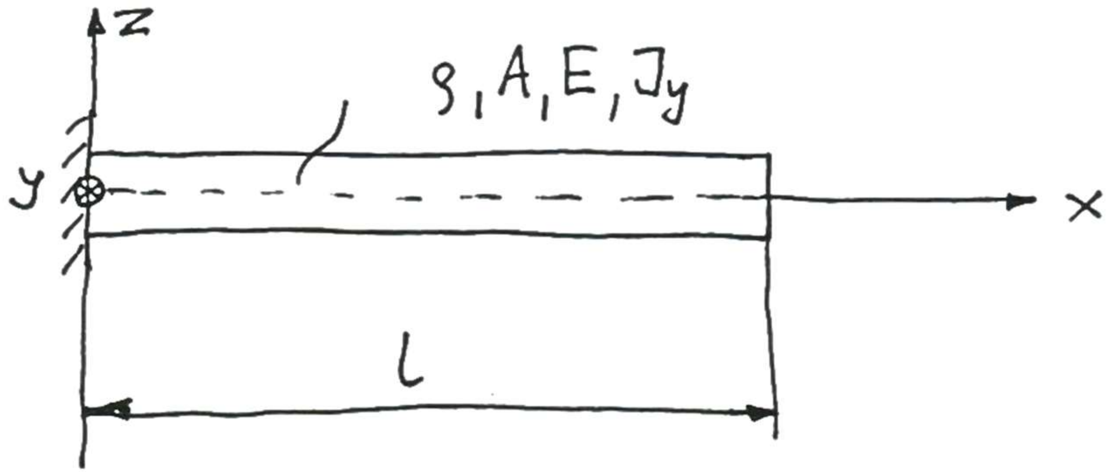
$$\left.\begin{aligned}
N_1(\xi) &= 1 - \frac{3}{l_e^2}\xi^2 + \frac{2}{l_e^3}\xi^3 \\
N_2(\xi) &= \xi - \frac{2}{l_e}\cdot\xi^2 + \frac{1}{l_e^2}\cdot\xi^3 \\
N_3(\xi) &= \frac{3}{l_e^2}\xi^2 - \frac{2}{l_e^3}\xi^3 \\
N_4(\xi) &= -\frac{1}{l_e}\xi^2 + \frac{1}{l_e^2}\xi^3
\end{aligned}\right\}\ \text{shape functions of a beam F.E.}$$

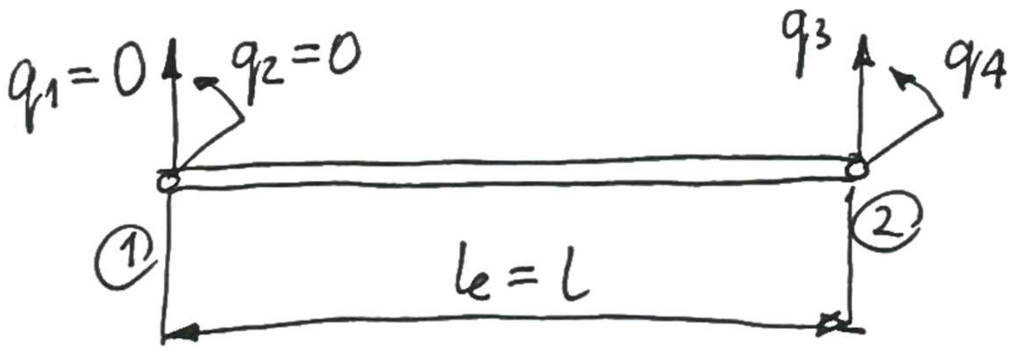
$$\Rightarrow\ T_e = \frac{1}{2}\underset{1\times 4}{\lfloor \dot{q} \rfloor_e}\cdot\underset{4\times 4}{[m]_e}\cdot\underset{4\times 1}{\{\dot{q}\}_e}$$

consistent mass matrix of a beam FE.

$$[m]_e = \frac{\rho A l_e}{420}\begin{bmatrix} 156 & 22l_e & 54 & -13l_e \\ 22l_e & 4l_e^2 & 13l_e & -3l_e \\ 54 & 13l_e & 156 & -22l_e \\ -13l_e & -3l_e & -22l_e & 4l_e^2 \end{bmatrix}$$

EXAMPLE : FIND NATURAL FREQUENCIES AND VIBRATION MODES IN PLANE XZ FOR A CANTILEVER BEAM





$$\left(\underset{4\times4}{[K]} - \omega^2 \underset{4\times4}{[M]}\right)\underset{4\times1}{\{q\}} = \underset{4\times1}{\{0\}}$$

$$\left(\frac{2EJ_y}{l^3}\begin{bmatrix} 6 & 3l & -6 & 3l \\ 3l & 2l^2 & -3l & l^2 \\ -6 & -3l & 6 & -3l \\ 3l & l^2 & -3l & 2l^2 \end{bmatrix} - \frac{\omega^2 \varrho A l}{420}\begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l \\ 54 & 13l & 156 & -22l \\ -13l & -3l & -22l & 4l^2 \end{bmatrix}\right)\cdot \underset{4\times1}{\{q\}} = \underset{4\times1}{\{0\}}$$

a new constant: $\lambda = \dfrac{\varrho A l^4}{840\, EJ_y}\,\omega^2$, $q_1 = 0$, $q_2 = 0$

$$\left(\begin{bmatrix} 6 & -3l \\ -3l & 2l^2 \end{bmatrix} - \lambda \begin{bmatrix} 156 & -22l \\ -22l & 4l^2 \end{bmatrix}\right)\cdot \begin{Bmatrix} q_3 \\ q_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\det\left(\begin{bmatrix} 6-156\lambda & (22\lambda-3)\cdot l \\ (22\lambda-3)\cdot l & (2-4\lambda)\cdot l^2 \end{bmatrix}\right) = 0$$

$$(6-156\lambda)(2-4\lambda)l^2 - (22\lambda-3)^2 l^2 = 0$$

$$(12 - 24\lambda - 312\lambda + 624\lambda^2)\, l^2 - 484\lambda^2 l^2 + 132\lambda l^2 - 9l^2 = 0$$

$$140\lambda^2 - 204\lambda + 3 = 0 \quad \Rightarrow$$

$$\lambda_1 = 1.4857\cdot 10^{-2} \quad , \quad \lambda_2 = 1.4423$$

VIBRATION MODE	FREQUENCY FE MODEL (1FE) $f_i \Big/ \left(\sqrt{\frac{EJ_y}{\rho A L^4}}\right)$	 ANALYTICAL $\bar{f}_i \Big/ \left(\sqrt{\frac{EJ_y}{\rho A L^4}}\right)$	RELATIVE ERROR $\frac{\lvert f_i - \bar{f}_i \rvert}{f_i}$
1st	0.5626	0.5598	0.5 %
2nd	5.543	3.5087	58 %

$$\begin{bmatrix} 6 - 156\lambda_i & (22\lambda_i - 3)l \\ (22\lambda_i - 3)l & (2 - 4\lambda_i)l^2 \end{bmatrix} \cdot \begin{Bmatrix} q_3 \\ q_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

two linearly dependent equations

1st:

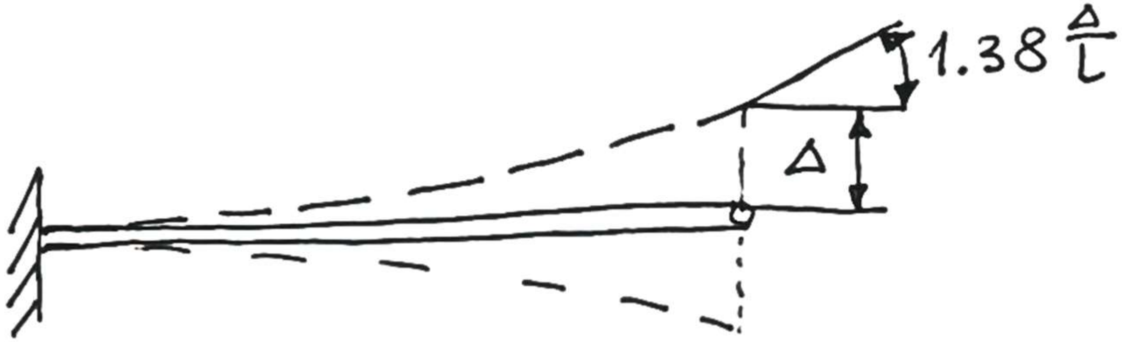
$$(6 - 156\lambda_i)\, q_3 + (22\lambda_i - 3)\, l \cdot q_4 = 0$$

$$q_4 = \frac{156\lambda_i - 6}{22\lambda_i - 3} \cdot \frac{q_3}{l}$$

if $q_3 = \Delta$ then: $q_4(\lambda_1) = 1.38 \cdot \frac{\Delta}{l}$

$$q_4(\lambda_2) = 7.62 \cdot \frac{\Delta}{l}$$

1st mode



2nd mode